# Useful Formulas

- $\sin^2(t)+\cos^2(t)=1$
- $\sin^2(t)=\frac{1-\cos(2t)}{2}$
- $\cos^2(t)=\frac{1+\cos(2t)}{2}$
- $e^{\ln(x)}=x$
- $\ln(e^x)=x$
- $\ln(a^b)=b\ln(a)$

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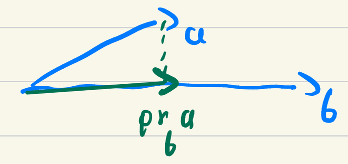
## Ch 11:

- $\vec{PQ}=Q-P$
- $\vec{a}=(a_1,a_2,a_3) \Rightarrow \|a\|=\sqrt{a_1^2+a_2^2+a_3^2}$
- $\vec{a}\parallel\vec{b} \Leftrightarrow \vec{a}=c\vec{b} \Leftrightarrow \vec{a}\times\vec{b}=\vec{0}$   parallel
- $\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b}=0$

**Dot Product:**
- $\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\cdot\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=a_1b_1+a_2b_2+a_3b_3$
- $a\cdot b=\|a\|\,\|b\|\cos(\theta)$

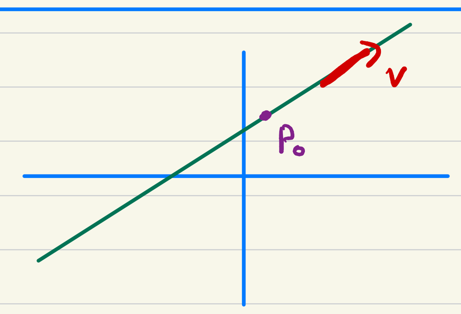
**Projection:**




$pr_b a=\left(\frac{a\cdot b}{\|b\|^2}\right)b$

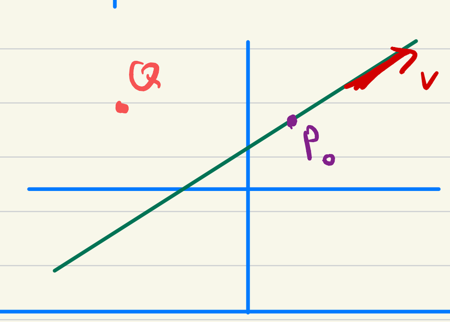
**Cross Product:**
- $a\times b=\begin{vmatrix} i & j & k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$
- $\|a\times b\|=\|a\|\,\|b\|\sin(\theta)$
- $\|a\times b\|=$ area of parallelogram formed by $a, b$
  Area of triangle $=\frac{1}{2}\|a\times b\|$
- $|a\cdot(b\times c)|=$ volume of parallelipiped formed by $a,b,c$

**Lines:**



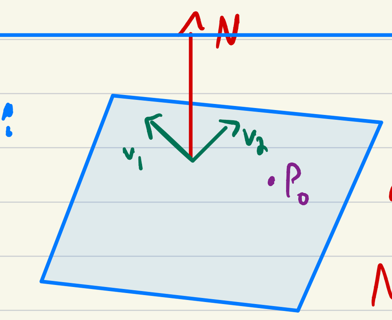

| Parametric | Vector | Symmetric |
|---|---|---|
| $x=x_0+v_1t$, $y=y_0+v_2t$, $z=z_0+v_3t$ ($P_0$, $v$) | $r(t)=P_0+vt$ $=\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}t$ | $\frac{x-x_0}{v_1}=\frac{y-y_0}{v_2}=\frac{z-z_0}{v_3}$ |

**Distance:**




$dist(Q,\text{line})=\frac{\|v\times\vec{QP_0}\|}{\|v\|}$

**Planes:**

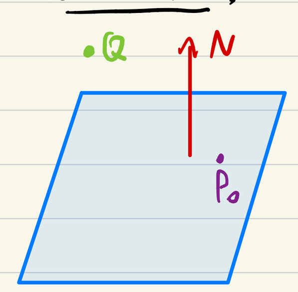



$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$N=(a,b,c)\perp$ plane
$P_0=(x_0,y_0,z_0)$ pt in plane

If $v_1,v_2$ parallel to plane then $N=v_1\times v_2$

**Distance:**




$dist(Q,\text{plane})=\frac{|N\cdot\vec{QP_0}|}{\|N\|}$

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## Ch 12:

- $r(t)=$ position
- $v(t)=r'(t)=$ velocity
- $a(t)=r''(t)=v'(t)=$ acceleration
- $T(t)=\frac{v(t)}{\|v(t)\|}=\frac{r'(t)}{\|r'(t)\|}$   unit tangent vector
- $N(t)=\frac{T'(t)}{\|T'(t)\|}$   unit normal vector
- $r(t)=$ smooth on interval $I$ if $\begin{cases} r'(t) \text{ cont on } I\\ r'(t)\neq 0 \text{ on interior of } I\end{cases}$
- $r(t)=$ piecewise smooth if it is smooth on a finite number of subintervals of $I$
- $L=\int_a^b\|r'(t)\|\,dt=$ length
- $a=a_T T+a_N N$ where $a_T=\frac{d\|v\|}{dt}=\frac{v\cdot a}{\|v\|}=$ tangential component

  $a_N=\|v\|\cdot\left\|\frac{dT}{dt}\right\|=\frac{\|v\times a\|}{\|v\|}=$ normal component

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- $\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2+y^2}$
  - $x=0,\ y\to0 \Rightarrow \lim_{y\to0}\frac{0y^2}{0+y^3}=0$
  - $x=y\to0 \Rightarrow \lim_{(x,y)\to(0,0)}\frac{x^4}{2x^4}=\frac{1}{2}\neq0$
  
  $\Rightarrow$ limit DNE

- $\lim_{(x,y)\to(0,0)}\frac{x^4y^2}{x^2+y^2}=\lim_{r\to0}\frac{r^4\cos^4(\theta)\,r^2\sin^2(\theta)}{r^2}=\lim_{r\to0}r^4\cos^4(\theta)\sin^2(\theta)$
  - $0\le|r^4\cos^4(\theta)\sin^2(\theta)|\le|r^4|$; $0 \to 0$ by squeeze, $\to 0$

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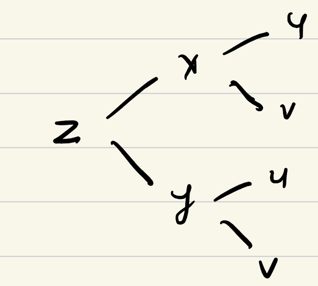
**Th:** If $f$ is "nice" (not always!) $f_{xy}=f_{yx}$

- $z=x^2+y^2$, $\begin{cases}x=3u+v\\ y=u-v\end{cases}$




$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$
$=2x\cdot3+2y\cdot1$
$=6\cdot(3u+v)+2\cdot(u-v)$

- $\nabla f=\begin{bmatrix}f_x\\f_y\\f_z\end{bmatrix}$
- $D_uf=\nabla f\cdot\frac{u}{\|u\|}=$ directional derivative

**Properties:**
(1) $D_uf$ maximizes at $u=\nabla f$ and its max is $\|\nabla f\|$
(2) $D_uf$ minimizes at $u=-\nabla f$ and its min is $-\|\nabla f\|$
(3) $D_uf=0$ when $u\perp\nabla f$

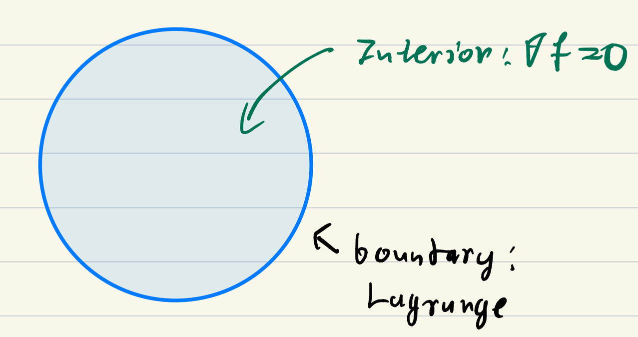
**Tangent Plane:**
(1) If $f(x,y,z)=c$ level set then $N=\nabla f$ normal
e.g. $x^2+y^2+z^2=4$ sphere then $N=\nabla f=(2x,2y,2z)=$ normal
(2) If $z=f(x,y)$ surface then take $g=f(x,y)-z$ and $N=\nabla g=$ normal
e.g. $f(x,y)=5x^2+5y^2$ then $g=5x^2+5y^2-z$ and $N=\nabla g=(10x,10y,-1)=$ normal

**Tangential Approximation:** $f(x_0+h,y_0+k)\approx f(x_0,y_0)+f_x(x_0,y_0)h+f_y(x_0,y_0)k$

**2nd Derivative test:** critical pts $\to \nabla f=0$, $\nabla f=$ DNE

If $P=$ crit pt then take $H=\begin{bmatrix}f_{xx} & f_{xy}\\ f_{xy} & f_{yy}\end{bmatrix}$
- $\det(H)<0$ saddle
- $\det(H)=0$ undeterm.
- $\det(H)>0$: $f_{xx}>0$ local min; $f_{xx}<0$ local max

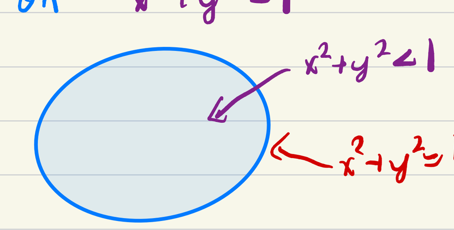
**Lagrange** (for extreme values)




For boundary: $\begin{cases}\nabla f=\lambda\nabla g\\ \text{boundary condition}\end{cases}$

e.g. $f(x,y)=x^4+y^2$ on $x^2+y^2\le1$

Then

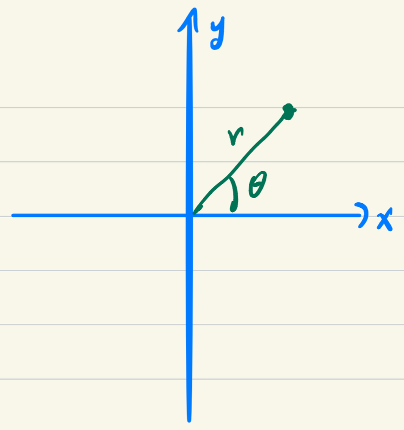


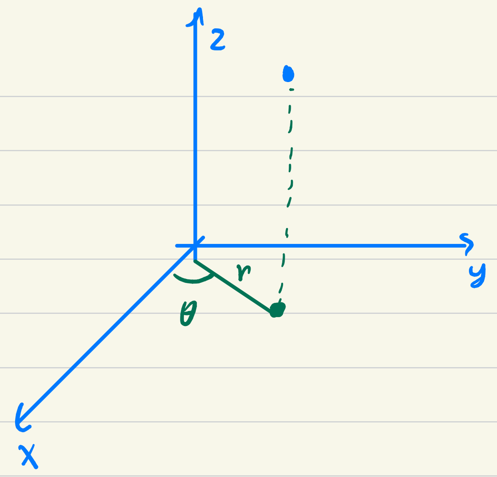

On interior: $\nabla f=\begin{bmatrix}4x^3\\2y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\Rightarrow Q=(0,0)$ crit pt

On boundary: $g(x,y)=x^2+y^2 \rightsquigarrow \nabla g=\begin{bmatrix}2x\\2y\end{bmatrix}$, $f(x,y)=x^4+y^2 \rightsquigarrow \nabla f=\begin{bmatrix}4x^3\\2y\end{bmatrix}$ $\Rightarrow$ Lagrange: $\begin{cases}4x^3=2\lambda x\\ 2y=2\lambda y\\ x^2+y^2=1\end{cases}$

## Ch 14:

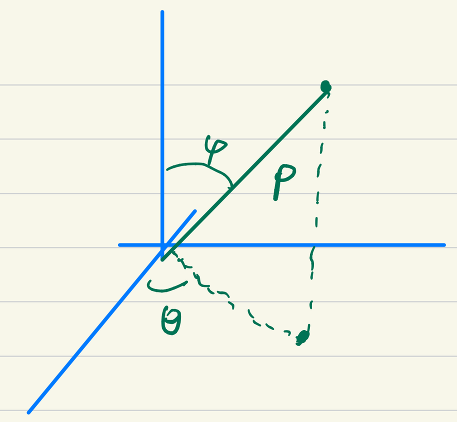
**Polar**




$x = r\cos(\theta)$
$y = r\sin(\theta)$
$z = z$
- $\boxed{x^2+y^2=r^2}$
- $\boxed{\iint f(x,y) = \iint f(r,\theta)\cdot r}$

**Spherical:**




$x = \rho\cos(\theta)\sin(\varphi)$
$y = \rho\sin(\theta)\sin(\varphi)$
$z = \rho\cos(\varphi)$

- $\boxed{\begin{array}{l} x^2+y^2+z^2=\rho^2 \\ x^2+y^2 = \rho^2\sin^2(\varphi) \end{array}}$
- $\boxed{\iint f(x,y,z) = \iint f(\theta,\varphi,\rho)\cdot\rho^2\sin(\varphi)}$

- $V = \iiint 1 \; dz\,dx\,dy$ volume of region
  $S = \iint_R \|r_x \times r_y\| \; dA$ surface area

**Change of variables:**

$x = x(u,v)$, $y = y(u,v)$ $\rightsquigarrow$ $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ or $u = u(x,y)$, $v = v(x,y)$ $\rightsquigarrow$ $J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$\iint_R f(x,y)\,dA = \iint_{R'} f(u,v)\cdot|\det(J)| \; dA'$$

$$\iint_R f(x,y)\,dA = \iint_{R'} f(u,v)\cdot\left|\frac{1}{\det(J')}\right| \; dA'$$

! Not much to memorize, Practice!!

## Ch 15:

$\int_C f\,ds = \int_a^b f(r(t))\cdot\|\frac{dr}{dt}\|\;dt$ line integral

$\iint_\Sigma f\,ds = \iint_R f(r(x,y))\cdot\|r_x \times r_y\| \; dA$ surface integral

$\int_C F\cdot dr =$
- $\int_a^b F(r(t))\cdot\frac{dr}{dt}\;dt$
- $f(\text{end pt}) - f(\text{start pt})$ if $F = \nabla f$ $(\text{curl } F) = 0$ (FTLI)
- $\pm\iint_R N_x - M_y \; dA$ if $C$ = closed, $F = \begin{bmatrix} M \\ N \end{bmatrix}$ (Green's) + = anti clock, − = clock
- $\pm\iint_\Sigma \text{curl}(F)\cdot n\;ds$ if $C$ = closed, $F = \begin{bmatrix} M \\ N \\ P \end{bmatrix}$ (Stokes) +: induced orient = orient of $C$

both ways

$\iint_\Sigma F\cdot n\;ds =$
- $\pm\iint_R F(r(x,y))\cdot(r_x \times r_y)\;dA$ +: induced orient = orient of exercise
- $\pm\iiint \text{div } F \; dV$ (Divergence) + = outwards (if $\Sigma$ = closed!)

**Parametrization:**

Standard: $r(x,y) = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$, $r_x \times r_y = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$, $\|r_x \times r_y\| = \sqrt{1+f_x^2+f_y^2}$

Cylinder ($R$ = fixed): $r(\theta,z) = \begin{bmatrix} R\cos(\theta) \\ R\sin(\theta) \\ z \end{bmatrix}$, $\|r_\theta \times r_z\| = R$

disk ($z$ = fixed): $r(r,\theta) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ 0 \end{bmatrix}$, $\|r_r \times r_\theta\| = R$

Cone ($\varphi$ = fixed): $r(\rho,\theta) = \begin{bmatrix} \rho\cos(\theta)\sin(\phi) \\ \rho\sin(\theta)\sin(\phi) \\ \rho\cos(\phi) \end{bmatrix}$, $r_\rho \times r_\theta = \rho\sin\phi$

Sphere: ($\rho$ = fixed) $r(\theta,\varphi) = \begin{bmatrix} R\cos(\theta)\sin(\varphi) \\ R\sin(\theta)\sin(\varphi) \\ R\cos(\varphi) \end{bmatrix}$, $\|r_\theta \times r_\varphi\| = R^2\sin(\varphi)$